Comment on " Regional Versus Global Entanglement in Resonating-Valence-Bond States"

In a recent Letter [1], Chandran and coworkers study the entanglement properties of valence bond (VB) states. Their main result is that VB states do not contain (or only an insignificant amount of) two-site entanglement, whereas they possess multi-body entanglement. Two examples ("RVB gas and liquid") are given to illustrate this claim, which essentially comes from a lower bound derived for spin correlators in VB states. While we do not question that two-site entanglement is generically "small" for isotropic VB states, we show in this Comment that (i) for the "RVB liquid" on the square lattice, the calculations and conclusions of Ref. [1] are incorrect. (ii) A simple analytical calculation gives the exact value of the correlator for the "RVB gas", showing that the bound found in Ref.1 is tight. (iii) The lower bound for spin correlators in VB states is equivalent to a celebrated result of Anderson dating from more than 50 years ago.

The $SU(2)$ symmetry of VB states guarantees that any two-spin reduced density matrix is a "Werner state" fully characterized by a parameter p. The considered pair of spins is entangled if $p > 1/3$. Chandran *et al.* used quantum information concepts such as monogamy of entanglement and quantum telecloning to obtain bounds on p. The number p is simply related to the correlator $\langle \mathbf{S}_i.\mathbf{S}_j\rangle$ between these two spins 1/2 ($\mathbf{S} = \sigma/2$, with σ Pauli matrices). We have $\langle \mathbf{S}_i.\mathbf{S}_j\rangle = -3/4p$ (and not "exactly equal to the parameter p" as stated in Ref. [1]).

(i) The "RVB liquid" is the equal amplitude superposition of all nearest-neighbour (NN) VB coverings of a bipartite lattice. Exact results can be obtained for small sizes L of the square $L \times L$ lattice. For $L = 4$, we do not recover the value $p \simeq 0.2004$ of Ref. [1], but find $p = 0.4457579115872$ for periodic boundary conditions (BC) and $p = 0.2281115037$ in the interior of a sample with open BC. However, what really matters is the behavior for large L. Exact calculations are difficult in this case, but Monte Carlo calculations are possible [2]. We computed the NN correlator $\langle \mathbf{S}_i.\mathbf{S}_j\rangle$ for large samples (up to $L = 128$) on the square lattice with periodic BC. The data of Fig. 1 shows that p is larger than 1/3 in the thermodynamic limit (we find $p = 0.3946(3)$, resulting in an entanglement of formation of $\simeq 0.0215$). Therefore, the "RVB liquid" on the square lattice *does* possess two-site (NN) entanglement, contrary to the claim of Ref. [1].

(ii) The "RVB gas" is the equal amplitude superposition of all *bipartite* VB coverings of a bipartite lattice. This is in fact the projection into the singlet sector of the (magnetically ordered) *Néel state* on this lattice. This observation can be used to calculate p exactly. The total spins S_A and S_B on sublattices A and B are maximal, couple antiferromagnetically and form a singlet (total spin $S = 0$). For a system of $2N$ spins, $S_A = S_B = N/2$. One then easily obtains that $\langle \mathbf{S}_i.\mathbf{S}_j\rangle = -1/4 - 1/(2N)$ if i and j belong to different sublattices. The equivalent exact result $p = 1/3 + 2/(3N)$ shows that the telecloning bound $p \leq 1/3 + 2/(3N)$ is tight. Two-site entanglement is therefore present in any finite "RVB gas" and vanishes only in the thermodynamic limit.

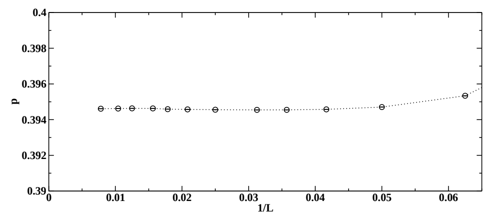

FIG. 1: Werner parameter p as a function of inverse linear system size $1/L$ for the square lattice "RVB liquid".

(iii) The telecloning bound on p in Ref. [1] reproduces an inequality found by Anderson [3], who derived a lower bound for the energy of antiferromagnetic spin models. Take a spin at site i, separated by any distance from a number z of symmetry-equivalent spins j: $\langle \mathbf{S}_i.\mathbf{S}_j\rangle$ (as well as p) is identical for all z spins at sites j. In this case, the telecloning bound is $p \leq 1/3 + 2/(3z)$ or equivalently $\langle \mathbf{S}_i.\mathbf{S}_j\rangle \geq -1/4 - 1/(2z)$, the result derived by Anderson. His result (of variational nature) on correlators in a given state is very general: it holds also for states other than singlets, is independent of any Hamiltonian and can be refined further (see e.g. Ref. [4]).

In conclusion, the bound obtained with quantum information techniques [1] has been familiar in the condensed matter context for a long time. Nevertheless, it is interesting to see that it can be derived in a totally different framework. For the two examples chosen in Ref. 1, typical condensed matter methods allowed us to provide in one case an exact solution, and to show that the results of Ref. 1 are incorrect in the other one.

We thank M. Mambrini for useful discussions, and D. Kaszlikowski and V. Vedral for correspondence.

Fabien Alet, Daniel Braun
Laboratoire de Physique Théorique, UMR CNRS 5152, Université Paul Sabatier, 31062 Toulouse, France
Grégoire Misguich
Institut de Physique Théorique, URA CNRS 2306, CEA Saclay, 91191 Gif sur Yvette, France

[1] A. Chandran *et al.*, Phys. Rev. Lett. **99**, 170502 (2007).
[2] We generalized the algorithm of A.W. Sandvik and R. Moessner, Phys. Rev. B **73**, 144504 (2006) to account for VB overlap properties.
[3] P.W. Anderson, Phys. Rev. **83**, 1260 (1951).
[4] R. Tarrach and R. Valentí, Phys. Rev. B **41**, 9611 (1990).